

The limiting behavior of some infinitely divisible exponential dispersion models

Shaul K. Bar-Lev* and Gérard Letac†

May 19, 2010

Abstract

Consider an exponential dispersion model (EDM) generated by a probability μ on $[0, \infty)$ which is infinitely divisible with an unbounded Lévy measure ν . The Jorgensen set (i.e., the dispersion parameter space) is then \mathbb{R}^+ , in which case the EDM is characterized by two parameters: θ_0 the natural parameter of the associated natural exponential family and the Jorgensen (or dispersion) parameter t . Denote by $EDM(\theta_0, t)$ the corresponding distribution and let Y_t is a r.v. with distribution $EDM(\theta_0, t)$. Then if $\nu((x, \infty)) \sim -\ell \log x$ around zero we prove that the limiting law F_0 of Y_t^{-t} as $t \rightarrow 0$ is of a Pareto type (not depending on θ_0) with the form $F_0(u) = 0$ for $u < 1$ and $1 - u^{-\ell}$ for $u \geq 1$. Such a result enables an approximation of the distribution of Y_t for relatively small values of the dispersion parameter of the corresponding EDM. Illustrative examples are provided.

Keywords: Exponential dispersion model; infinitely divisible distributions; limiting distributions; natural exponential family.

2000 Mathematics Subject Classification: 62E20; 60E07.

*Department of Statistics, University of Haifa, Haifa 31905, Israel (email: bar-lev@stat.Haifa.ac.il)

†Laboratoire de Statistique et Probabilités, Université Paul Sabatier, 31062 Toulouse, France (letac@cict.fr)

1 Introduction

Let $\{F_t : 0 < t < t_0 \leq \infty\}$ be a family of distributions associated with positive r.v.'s $\{Y_t\}$ and Laplace transforms (LT's) $L_t(u) = E(e^{-uY_t})$. Also let Y_0 be a r.v. with distribution F_0 . Bar-Lev and Enis (1987, Theorem 1) showed that $Y_t^{-t} \xrightarrow{D} Y_0$ as $t \rightarrow 0$ (where \xrightarrow{D} designates a convergence in distribution) iff

$$\lim_{t \rightarrow 0} L_t(u^{1/t}) = \bar{F}_0(u) = 1 - F_0(u) \quad (1)$$

at all continuity points of F_0 . As Bar-Lev and Enis indicated, such a result can be viewed as a "centralization" problem in the following sense. In many cases the limiting distribution of Y_t as $t \rightarrow 0$ is degenerate. A measurable transformation $g_t(Y_t)$ is then sought whose limiting distribution is non-degenerate. Accordingly, their Theorem 1 suggests a consideration of $g_t(Y_t) = Y_t^{-t}$ (or, equivalently, of $-t \ln Y_t$) whose limiting distribution is non-degenerate (provided that (1) is satisfied). Bar-Lev and Enis presented several examples which satisfy (1). However, these examples heavily depend on the explicit (and relatively 'nice') form of L_t .

A natural question then arises: Can one delineate subclasses of distributions which satisfy (1), regardless of the explicit form of L_t ? Indeed, in this note we provide such a subclass, namely a subclass of exponential dispersion models (EDM's) generated by a probability μ on $[0, \infty)$ which is infinitely divisible of type 1 (c.f., Jorgensen, 1987, 1997, 2006, and Letac and Mora, 1990). For such a subclass, the EDM is characterized by two parameters: θ_0 the natural parameter of the associated natural exponential family (NEF) and the Jorgensen (or, equivalently, the dispersion parameter) $t \in \mathbb{R}^+$. We denote such a subclass of distributions by $EDM(\theta_0, t)$ and prove that if Y_t has a distribution in $EDM(\theta_0, t)$ then $Y_t^{-t} \xrightarrow{D} Y_0$ as $t \rightarrow 0$, where the distribution F_0 of Y_0 is of a Pareto type (not depending on θ_0) with the form

$$F_0(u) = \begin{cases} 0, & \text{if } u < 1, \\ 1 - u^{-\ell}, & \text{if } u \geq 1, \end{cases}$$

for some $\ell > 0$. Such a result enables an approximation of the distribution of Y_t for relatively small values of the dispersion parameter of the corresponding EDM.

This note is organized as follows. In Section 2 we first introduce some preliminaries on NEF's and EDM's and then present our main result. Section 3 is devoted to some examples.

2 Preliminaries and the main result

We first briefly introduce some preliminaries related to NEF's and their associated EDM's. Let μ be a probability measure on \mathbb{R} . Assume that the effective domain of μ has a nonempty interior, i.e.,

$$\Theta \doteq \text{int } D = \left\{ \theta \in \mathbb{R} : L(\theta) = \int_{\mathbb{R}} e^{-\theta x} \mu(dx) < \infty \right\} \neq \emptyset.$$

The NEF generated by μ is then given by the set of probabilities

$$\left\{ P(\theta, \mu)(dx) = \frac{e^{-\theta x}}{L(\theta)} \mu(dx), \theta \in D \right\}.$$

Note that since μ is a probability measure then $0 \in D$. The Jorgensen set is defined by

$$\Lambda = \{t \in \mathbb{R}^+ : L^t \text{ is a LT of some measure } \mu_t \text{ on } \mathbb{R}\},$$

whereas the corresponding EDM is the class of probabilities

$$\left\{ P(\theta, t, \mu_t)(dx) = \frac{e^{-\theta x}}{L^t(\theta)} \mu_t(dx), \theta \in D, t \in \Lambda \right\}, \quad (2)$$

Note that the class of EDM's is abundant since any probability measure with a LT generates an EDM. An EDM is therefore parameterized by the two parameters $(\theta, t) \in D \times \Lambda$, where θ is the natural parameter of the corresponding NEF and t is termed the dispersion parameter. EDM's have a variety of applications in various areas, in particular, in generalized linear models (replacing the normal model as the error model distribution) and actuarial studies. Note that if μ is infinitely divisible then the Jorgensen set (or, equivalently, the dispersion parameter space) is $\Lambda = \mathbb{R}^+$. Also note that if Y_t is a r.v. with distribution in (2) then its LT is given by

$$E(e^{-sY_t}) = \left(\frac{L(\theta + s)}{L(\theta)} \right)^t. \quad (3)$$

Now we consider the case where μ is infinitely divisible law of type 1 concentrated on \mathbb{R}^+ . By this we mean that there exists an unbounded positive measure ν on $(0, \infty)$ such that for $\theta \geq 0$ one has $L(\theta) = \int_0^\infty e^{-\theta x} \mu(dx) \doteq e^{k(\theta)}$ with $k(\theta) = -\int_0^\infty (1 - e^{-\theta x}) \nu(dx)$ and $\int_0^\infty \min(1, x) \nu(dx) < \infty$. Therefore

ν is the Lévy measure of μ . The Lévy process associated with such a μ is sometimes called a pure jump subordinator (in this respect, of Lévy measures for NEF's, see also Kokonendji and Khoudar, 2006). Note that this implies that $\lim_{\theta \rightarrow \infty} k(\theta) = -\infty$ since ν is unbounded and therefore $\lim_{\theta \rightarrow \infty} L(\theta) = 0$ and $\mu(\{0\}) = 0$. We are now ready to present our main result relating to the limiting distribution of Y_t^{-t} as $t \rightarrow 0$.

Proposition 1 *Assume that μ is an infinitely divisible probability measure of type 1. Also assume that $G(x) \doteq \int_{x+}^{\infty} \nu(dy) = \nu((x, \infty))$ is such that*

$$\lim_{x \rightarrow 0} G(x)/\log x = -\ell$$

for some $\ell > 0$. Let $\theta_0 \geq 0$, then

$$\lim_{t \rightarrow 0} \left(\frac{L(\theta_0 + u^{1/t})}{L(\theta_0)} \right)^t = \begin{cases} 1, & \text{if } u < 1, \\ u^{-\ell}, & \text{if } u \geq 1, \end{cases} \quad (4)$$

implying by (1) that $Y_t^{-t} \xrightarrow{D} Y_0$ as $t \rightarrow 0$, where the distribution F_0 of Y_0 is given by

$$F_0(u) = \begin{cases} 0, & \text{if } u < 1, \\ 1 - u^{-\ell}, & \text{if } u \geq 1, \end{cases}$$

Proof. We first prove (4) for $\theta_0 = 0$. For this, we give another presentation of k in terms of G which is obtained by an integration by parts. For $\epsilon > 0$ consider the Stieltjes integral

$$k_{\epsilon}(\theta) = - \int_{\epsilon+}^{\infty} (1 - e^{-\theta x}) \nu(dx) = (e^{-\theta \epsilon} - 1)G(\epsilon) - \theta \int_{\epsilon}^{\infty} e^{-\theta x} G(x) dx. \quad (5)$$

Since for $\epsilon \rightarrow 0$ we have $(e^{-\theta \epsilon} - 1) \sim -\theta \epsilon$ and $G(\epsilon) \sim -\ell \log \epsilon$ we get

$$k(\theta) = \lim_{\epsilon \rightarrow 0} k_{\epsilon}(\theta) = -\theta \int_0^{\infty} e^{-\theta x} G(x) dx. \quad (6)$$

If $0 < u < 1$ we have $\lim_{t \rightarrow 0} u^{1/t} = 0$. Since μ is a probability $L(0) = 1$ and thus $\lim_{t \rightarrow 0} L(u^{1/t})^t = 1$. If $u = 1$ we also have that $\lim_{t \rightarrow 0} L(1)^t = 1$. If $u > 1$, We fix an arbitrary $0 < \epsilon < \ell$. By the definition of ℓ there exists

$0 < \eta < 1$ such that if $0 < x < \eta$ then $-(\ell - \epsilon) \log x < G(x) < -(\ell + \epsilon) \log x$. We now use (6) for writing

$$\left| k(\theta) + \theta \int_{\eta}^{\infty} e^{-\theta x} G(x) dx - \ell \theta \int_0^{\eta} e^{-\theta x} \log x dx \right| < -\epsilon \theta \int_0^{\eta} e^{-\theta x} \log x dx \quad (7)$$

and observing that

$$\theta \int_{\eta}^{\infty} e^{-\theta x} G(x) dx \leq G(\eta) \theta \int_{\eta}^{\infty} e^{-\theta x} dx = G(\eta) e^{-\eta \theta} \xrightarrow{\theta \rightarrow \infty} 0. \quad (8)$$

We need now the following evaluation. For $\eta > 0$ we have

$$\lim_{\theta \rightarrow \infty} \frac{1}{\log \theta} \theta \int_0^{\eta} e^{-\theta x} \log x dx = -1. \quad (9)$$

To prove (9), we obtain, by a change of variable to $v = \theta x$, that

$$\frac{1}{\log \theta} \theta \int_0^{\eta} e^{-\theta x} \log x dx = \frac{1}{\log \theta} \int_0^{\eta \theta} e^{-v} \log v dv - \int_0^{\eta \theta} e^{-v} dv \xrightarrow{\theta \rightarrow \infty} 0 - 1, \quad (10)$$

where the last term on right hand side of (10) follows since $\int_0^{\infty} e^{-v} \log v dv$ converges and $\int_0^{\infty} e^{-v} dv = 1$. We now divide both sides of (7) by $\log \theta$ and let $\theta \rightarrow \infty$. From (8) and (9) we get that for all $\epsilon > 0$

$$-\ell - \epsilon \leq \liminf_{\theta \rightarrow \infty} \frac{1}{\log \theta} k(\theta) \leq \limsup_{\theta \rightarrow \infty} \frac{1}{\log \theta} k(\theta) \leq -\ell + \epsilon,$$

i.e., $\lim_{\theta \rightarrow \infty} \frac{1}{\log \theta} k(\theta) = -\ell$. Applying this to $\theta = u^{1/t}$ with a fixed $u > 1$ and letting $t \rightarrow 0$, we get

$$\lim_{t \rightarrow 0} t k(u^{1/t}) = -\ell \log u, \quad \lim_{t \rightarrow 0} (L(u^{1/t}))^t = \frac{1}{u^{\ell}},$$

which is the desired result. Finally suppose that $\theta_0 > 0$ and denote $k_{\theta_0}(\theta) = k(\theta_0 + \theta) - k(\theta_0)$. Trivially,

$$k_{\theta_0}(\theta) = - \int_0^{\infty} (1 - e^{-\theta x}) \nu_{\theta_0}(dx) \text{ with } \nu_{\theta_0}(dx) = e^{-\theta_0 x} \nu(dx)$$

and consider

$$\begin{aligned} G_{\theta_0}(x) &= \nu_{\theta_0}((x, \infty)) = \int_{x+}^{\infty} e^{-\theta_0 y} \nu(dy) \\ &= e^{-\theta_0 x} G(x) - \theta_0 \int_x^{\infty} e^{-\theta_0 y} G(y) dy \\ &= k_x(\theta_0) + G(x) \end{aligned} \quad (11)$$

$$= k_x(\theta_0) + G(x) \quad (12)$$

Line (11) is obtained by an integration by parts, whereas line (12) uses (5). We see easily from (12) that $\lim_{x \rightarrow 0} G_{\theta_0}(x)/\log x = -\ell$. Therefore we are in the same situation as in the proof of the first part with $\theta_0 = 0$, and thus the proof is completed. ■

The following remark is useful to obtain a genralization of the three examples presented in Section 3 for any $\ell > 0$.

Remark 1 Suppose that $(\mu_t)_{t>0}$ is a family of infinitely divisible distributions with $\mu_t * \mu_s = \mu_{t+s}$, where $t, s > 0$. Assume that μ_1 fulfills the premises of Proposition 1 with $G(x) \sim -\ell \log x$. Then obviously for any fixed $t > 0$, μ_t also fulfills such premises with $t\ell$ replacing the role of ℓ . In all of the examples below, we have $\ell = 1$ and this remark shows how to get from them other examples with arbitrary $\ell > 0$.

3 Examples

Example 1 If $\mu(dx) = e^{-x}1_{(0,\infty)}(x)dx$ the corresponding infinitely divisible family is the gamma family with scale parameter 1. The Lévy measure here is $\nu(dx) = e^{-x}1_{(0,\infty)}(x)\frac{dx}{x}$ and $\ell = 1$.

Example 2 A discrete example is

$$\nu(dx) = \sum_{n=1}^{\infty} \delta_{1/n}.$$

We have $k(\theta) = \sum_{n=1}^{\infty} \frac{1}{n}(1 - e^{-\theta/n})$, $G(x) = \sum_{n=1}^{\lfloor 1/x \rfloor} 1/n \sim -\log x$ if $x \rightarrow 0$ and $\ell = 1$. The probability μ is the distribution of $\sum_{n=1}^{\infty} \frac{X_n}{n}$, where X_n is Poisson distributed with mean $1/n$ and the $(X_n)_{n=1}^{\infty}$ are independent.

Example 3 Utilizing Example 2.2 in [1], consider the infinitely divisible distribution μ on $(0, \infty)$ with Laplace transform defined on $\theta \geq 0$ given by $\theta + 1 - \sqrt{\theta^2 + 2\theta}$. Note that the densities of the corresponding EDM are Bessel densities related to a symmetric random walk (see Feller, 1971, pp. 60-61).

The related Lévy measure of μ is

$$\nu(dx) = {}_1F_1(1/2; 1; -2x)1_{(0,\infty)}(x)\frac{dx}{x}, \quad (13)$$

where ${}_1F_1(a; b; z)$ is the so called confluent entire function defined by $\sum_{n=0}^{\infty} \frac{(a)_n}{n!(b)_n} z^n$, where $(a)_0 = 1$ and $(a)_{n+1} = (a+n)(a)_n$ define the Pochhammer symbol $(a)_n$. Therefore ν has a density equivalent to $1/x$ when $x \rightarrow 0$ which implies that $G(x) \sim -\log x$. Proposition 1 is thus satisfied with $\ell = 1$. To check the correctness of (13) observe that if $k(\theta) = \log(\theta + 1 - \sqrt{\theta^2 + 2\theta})$ then

$$k'(\theta) = - \int_0^{\infty} e^{-\theta x} x \nu(dx) = - \frac{1}{\sqrt{\theta+2}} \frac{1}{\sqrt{\theta}} = - \int_0^{\infty} e^{-\theta x} f(x) dx \times \int_0^{\infty} e^{-\theta y} g(y) dy,$$

where

$$f(x) = e^{-2x} \frac{1}{\sqrt{\pi x}} \mathbf{1}_{(0,\infty)}(x), \quad g(y) = \frac{1}{\sqrt{\pi y}} \mathbf{1}_{(0,\infty)}(y).$$

Therefore the density of $x\nu(dx)$, $x > 0$, is given by the convolution product $f * g$, where by a change of variable $y = tx$ and employing a Taylor expansion, one gets

$$\int_0^x f(y)g(x-y)dy = \frac{1}{\pi} \int_0^1 \frac{e^{-2xt}}{\sqrt{t(1-t)}} dt = {}_1F_1(1/2; 1; -2x).$$

Let us fix $t > 0$. Recall (see [2]) that for $\theta > 0$ the function $(\theta + 1 - \sqrt{\theta^2 + 2\theta})^t$ is the Laplace transform of the density $f_t(x) = \frac{t}{x} e^{-x} I_t(x) \mathbf{1}_{(0,\infty)}(x)$ where the Bessel function $I_t(x)$ is

$$I_t(x) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n+1+t)} \frac{x^{2n+t}}{2^{n+t}}.$$

We fix now $\ell > 0$. If Y_t has density $f_{t\ell}$ this implies that the density of $U = Y_t^{-t}$ is

$$g_t(u) = \ell t^2 u^{2t+1} e^{-\frac{1}{u^t}} I_{t\ell}\left(\frac{1}{u^t}\right) \mathbf{1}_{(0,\infty)}(u).$$

It would be quite delicate to prove directly from this last formula that when $t \rightarrow 0$ the law $g_t(u)du$ converges to the Pareto law

$$\mathbf{1}_{(1,\infty)}(u) \frac{\ell du}{u^{\ell+1}}$$

as shown by our Proposition.

References

- [1] Bar Lev, S.K. and Enis, P. (1987). Existence of moments and an asymptotic result based on a mixture of exponential distributions. *Statistics and Probability Letters*, 5, 273–277.
- [2] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol. II, Second Edition, Wiley, New York.
- [3] Jorgensen, B. (1987). Exponential dispersion models (with discussion). *J. Roy. Statist. Soc. Ser. B*, 49, 127–162.
- [4] Jorgensen, B. (1997). *The Theory of Dispersion Models*. Chapman and Hall, London.
- [5] Jorgensen, B. (2006). Dispersion models. *Encyclopedia of Statistical Sciences*, New York, John Wiley.
- [6] Kokonendji, C.C. and Khoudar, M. (2006). On Lévy measures for infinitely divisible natural exponential families. *Statistics and Probability Letters*, 76, 1364–1368.
- [7] Letac, G. and Mora, M. (1990). Natural exponential families with cubic variance functions. *The Annals of Statistics*, 18, 1–37.